On Vector Calculus in Maxwell's Equations

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Introduction

During the early discoveries of electromagnetic theory, one the primary challenges, besides trying to understand electromagnetism, was mathematically modeling fundamental rules for the theory. It was much easier to model the universal laws by separating them into specific scenarios. For instance, we had theories of electricity, as well as theories of magnetism, but they were not combine them until 1755 by Henry Elles. Two mathematical tools they used to model electromagnetic laws were the curl and divergence operators. This paper will discuss some of the uses of divergence and curl in the generalized electromagnetic models, known as Maxwell's Equations.

Divergence

The divergence is an operator on a vector field, which in our case is the electromagnetic field. It takes a vector field and turns it into a scalar field. In short, the divergence is a measure of the expansion of a vector field at a point. The method of measurement is by taking the flux of an infinitesimal volume surrounding a point. The formal definition for divergence is the surface integral:

$$div(F)_p = \lim_{V \to \{p\}} \oiint_{\partial V} \frac{F \cdot \hat{n}}{|V|} \, dS$$

where p is the point where you're taking the divergence at, V is the space around p, F is the vector field, and $\hat{n}$ is the normal vector to V. In Cartesian coordinates, this can be further simplified to:

$$div(F) = \nabla \cdot \vec{F}$$

where ∇ is the differential operator. The divergence is probably best shown graphically as it represents the expansiveness of F.

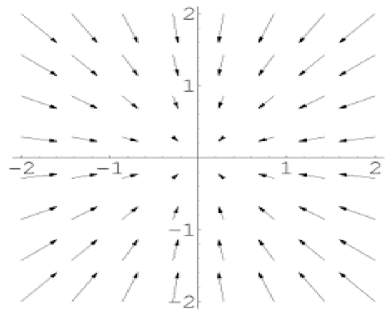

Figure 1: A vector field with negative divergence everywhere

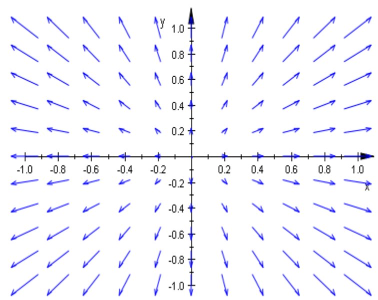

Figure 2: A vector field with positive divergence everywhere

As you can see in figure 1 the field is constantly shrinking, which indicates a negative divergence. In figure 2, the field constantly is expanding so the divergence is positive.

Example: Find the divergence at $p = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ of the vector field $\vec{F} = \left\langle 2x^2 + y, x + y - 2z, xz \right\rangle$

$$\text{div}(F)_p = \boldsymbol{\nabla} \cdot F(p) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 4x + 1 + x = 11$$

The divergence is most useful because of the divergence theorem which states that the flux of a vector field through a surface is equal to the volume integral of the divergence over the region inside the surface. Mathematically this is represented by the equation

$$\iiint_V \nabla \cdot \vec{F}\, dV = \oiint_{\partial V} F \cdot \hat{n}\, dS$$

The left side of the equation is the volume integral of the divergence, and the right side is the flux through the boundary of V. In electromagnetism this comes up as one way to represent Gauss's Law.

Gauss's Law

In electrostatics, the electric field of a point charge $\vec{E}$ can be calculated with the formula:

$$\vec{E} = \frac{k_e q}{r^2}$$

where q is the charge of the particle, r is the distance from the charge, and k_e is the electric field constant. The electric field is a vector field meaning we can use all of the normal vector field operations including divergence. The flux of the electric field Φ through a surface S is:

$$\Phi = \oint\!\!\!\oint_S E \cdot dA$$

where dA is an infinitesimally small surface area on S. Gauss's Law states that $\Phi = \frac{Q}{\epsilon_0}$ where Q is the charge enclosed by S and ϵ_0 is the permitivity of free space constant. By using the divergence theorem, we can transform Gauss's Law from an integral equation to a differential one:

$$\boldsymbol{\nabla} \cdot F = \frac{\rho}{\epsilon_0}$$

where ρ is the charge density.

Curl

Similar to the divergence, the curl is an operation on vector field which transforms it into another vector field. While divergence measures expansion, curl measures the amount of rotation. If you have a vector field you can imagine putting a water wheel at some point, and the direction and speed that it turns would be the curl.

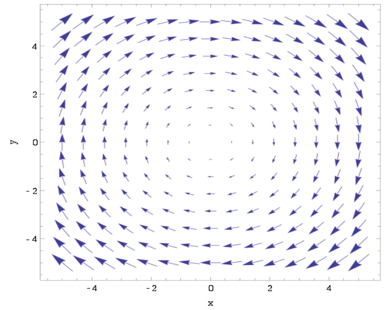

Figure 3: A vector field with uniform curl

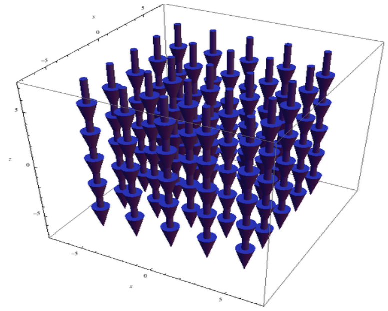

Figure 4: Curl of previous vector field

The definition of curl is more complicated than divergence but in 3-dimensions it is defined as:

$$curl(F) = \nabla \times F = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$

Example: Find the curl at $p = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ of the vector field $\vec{F} = \left\langle 2x^2 + y, x + y - 2z, xz \right\rangle$

First find the general curl then replace x, y, z with coordinates from p:

$$\text{curl}(F)_p = \nabla \times F = \langle (0) - (-2), (0) - (z), (1) - (1) \rangle = \langle 2, 3, 0 \rangle$$

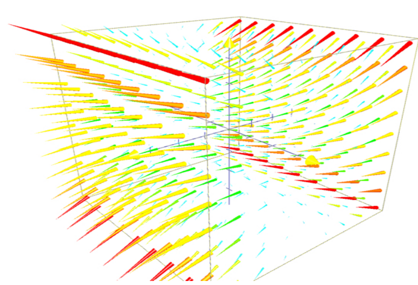

Figure 5: Vector Field $\vec{F}$

The sign of the curl is dependant on orientation or the right hand rule. For instance, in Figure 3, the vector field is rotating clockwise, so the curl is negative. The implications in electromagnetic theory of this field operation have mostly to do with the translation from electric fields to magnetic fields. One property that is important to this is the divergence of a curl is always 0, mathematically represented by the equation:

$$\nabla \cdot (\nabla \times F) = 0$$

While this won't be proved in this paper, it can be easily shown by seeing that the second derivatives are equal.

Gauss's Law of Magnetism

Gauss's Law of Magnetism states that the flux of a magnetic field B through any closed surface S is equal to 0, or

$$\oiint_S \boldsymbol{B} \cdot dA = 0$$

The meaning of this has huge implications in magnetism stating that a magnet must have both a north and a south pole otherwise one could find a surface such that the flux of the magnetic field isn't 0. Similarly to Gauss's Law of Electric Flux, we can turn this integral equation into a derivative equation with the divergence theorem. This new equation, $\nabla \cdot B = 0$, means that the magnetic field has divergence equal to 0 everywhere, meaning it is what is known as a solenoidal vector field. The Fundamental Theorem of Vector Calculus states that all smooth, decaying, 3-dimensional vector fields can be formed by adding a irrotational vector field and a solenoidal vector field. An irrotational vector field is one where the curl equals 0 everywhere. This means that the magnetic field has no irrotational component to it. Thus, all magnetic fields are the curl of some vector field $\boldsymbol{A}$, which is known as the magnetic field potential, and allows for easy translation between electric fields and magnetic fields.

References

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[2] Richard Feynmann and Matthew Sands. *The Feynman Lectures on Physics.* URL: `www.feynmanlectures.caltech.edu/`.

[3] Paul C. Matthews. *Vector Calculus.* Springer, 1998.